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Some Designs Which Admit Strong Tactical Decompositions

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Whenever there exist affine planes of orders $n - 1$ and n, a construction is given for a $2 - ((n+1)(n-1)^2, n(n-1), n)$ design admitting a strong tactical decomposition. These designs are neither symmetric nor strongly resolvable but can be embedded in symmetric $2 - (n^3 - n + 1, n^2, n)$ designs.

1. INTRODUCTION

If a $2 - (v, k, \lambda)$ design $\mathbf{D}$ admits a tactical decomposition with v_1 point classes and b_1 block classes then $b + v_1 \geqslant v + b_1$, (see [1]). Decompositions for which $b + v_1 = v + b_1$ are of special interest and are called strong. Any tactical decomposition of a symmetric design is strong. A strong tactical decomposition of a nonsymmetric design is called a strong resolution if it has only one point class (and the design is called strongly resolvable). Any affine design is strongly resolvable with the parallel classes giving the block classes of the strong resolution. Thus the study of designs which admit strong tactical decompositions gives a way of unifying similar theorems for symmetric and affine designs. One instance of this is in [2] where there is a proof of an orbit theorem for designs with strong tactical decompositions which includes as corollaries the orbit theorems for symmetric and affine designs. This is, we believe, the first unifying proof of these theorems.

The object of this paper is to show that, whenever there exist affine planes of orders $n - 1$ and n, there exist $2 - ((n+1)(n-1)^2, n(n-1), n)$ designs which admit strong tactical decompositions but are neither symmetric nor strongly resolvable. Thus, studying strong tactical decompositions does more than just study symmetric and strongly resolvable designs. The designs constructed can be embedded in symmetric $2 - (n^3 - n + 1, n^2, n)$ designs. This shows the existence of a symmetric $2 - (n^3 - n + 1, n^2, n)$ whenever there exist affine planes of orders $n - 1$

and n. The existence of symmetric designs with these parameters was known previously to R. Wilson.

All results used, unless otherwise stated, can be found in [3].

2. The Construction

If n is any integer such that there exist affine planes of orders $n - 1$ and n then we shall give a construction for a $2 - ((n + 1)(n - 1)^2, n(n - 1), n)$ which admits a strong tactical decomposition.

Let $\mathbf{A}_1, \ldots, \mathbf{A}_{n+1}$ be affine planes of order $n - 1$ defined on distinct point sets. For each α, $1 \leqslant \alpha \leqslant n + 1$, let $\mathbf{A}_{\alpha\beta}$, $(\beta = 1, \ldots, n)$, be the parallel classes of $\mathbf{A}_\alpha$ and let $a_{\alpha\beta\gamma}$, $(\gamma = 1, \ldots, n - 1)$, be the lines of $\mathbf{A}_{\alpha\beta}$. Let $\mathbf{B}_1, \ldots, \mathbf{B}_n$ be any affine planes of order n and let $\mathbf{B}_{\theta\phi}$, $(\phi = 1, \ldots, n + 1)$, be the parallel classes of $\mathbf{B}_\theta$. Now, for each fixed θ, let B_θ be a chosen point of $\mathbf{B}_\theta$ and let $b_{\theta\phi\mu}$, $(\mu = 1, \ldots, n - 1)$, be the lines of $\mathbf{B}_{\theta\phi}$ other than the (necessarily unique) one which contains B_θ. Finally for each fixed β let λ_β be any bijection from the lines of all the $\mathbf{A}_{\alpha\beta}$ onto the points of $\mathbf{B}_\beta \backslash B_\beta$ such that if $X_{\alpha\beta\gamma}$ denotes the image of $x_{\alpha\beta\gamma}$ under λ_β,

$$\text{for each } \alpha, \bigcup_{\gamma=1}^{n-1} X_{\alpha\beta\gamma} \cup B_\beta \text{ is a line of } \mathbf{B}_{\beta\alpha}. \tag{1}$$

Thus, essentially, for each fixed β we take the lines of the βth parallel classes of all the $\mathbf{A}_\alpha$ and associate them with the points of $\mathbf{B}_\beta \backslash B_\beta$ in any way provided that the parallel class of any given $\mathbf{A}_\alpha$ is associated with a line through B_β.

We now define an incidence structure, $\mathbf{D}(n)$, as follows. The points of $\mathbf{D}(n)$ are the points of all the $\mathbf{A}_\alpha$ and, for any θ, ϕ, μ $(1 \leqslant \theta \leqslant n, 1 \leqslant \phi \leqslant n + 1, 1 \leqslant \mu \leqslant n - 1)$ we define a block $z_{\theta\phi\mu} = \bigcup_{X_{\alpha\theta\gamma} \in b_{\theta\phi\mu}} x_{\alpha\theta\gamma}$. Thus the blocks of $\mathbf{D}(n)$ are the union of various lines of the $\mathbf{A}_\alpha$ and each block contains one line from exactly n of them.

Theorem 1. *$\mathbf{D}(n)$ is a $2 - ((n + 1)(n - 1)^2, n(n - 1), n)$ design.*

Proof. Clearly the blocks of $\mathbf{D}(n)$ are distinct as point sets. Since each $\mathbf{A}_\alpha$ has $(n - 1)^2$ points, $\mathbf{D}(n)$ has $(n + 1)(n - 1)^2$ points. Similarly, since each block of $\mathbf{D}(n)$ is the union of n lines from the $\mathbf{A}_\alpha$, each block has $n(n - 1)$ points on it. Thus we have only to show that any pair of points are on n common blocks. If X and Y are any two points then there are unique integers α and α' such that $X \in \mathbf{A}_\alpha$ and $Y \in \mathbf{A}_{\alpha'}$.

Case (*a*). $\alpha = \alpha'$.

There exist unique β, γ such that $XY = x_{\alpha\beta\gamma}$ and X, Y are on a common block of $\mathbf{D}(n)$ only if it contains each point of $x_{\alpha\beta\gamma}$. The point $X_{\alpha\beta\gamma}$ is on $n + 1$ lines of $\mathbf{B}_\beta$ but, since one of these contains B_β, it is on only n lines of the form $b_{\beta\phi\mu}$. Since there is a one-to-one correspondence between the lines $b_{\beta\phi\mu}$ containing $X_{\alpha\beta\gamma}$ and the blocks $z_{\beta\phi\mu}$ containing $x_{\alpha\beta\gamma}$, X and Y are on exactly n common blocks of $\mathbf{D}(n)$.

Case (*b*). $\alpha \neq \alpha'$.

For each β there exist unique γ and γ' such that $X \in x_{\alpha\beta\gamma}$ and $Y \in x_{\alpha'\beta\gamma'}$. The points $X_{\alpha\beta\gamma}$ and $X_{\alpha'\beta\gamma'}$ are joined by a unique line of $\mathbf{B}_\beta$ which, because of (1), does not contain B_β and is therefore of the form $b_{\beta\phi\mu}$. But X, Y are on the block $z_{\beta\phi\mu}$ if and only if $X_{\alpha\beta\gamma}$, $X_{\alpha'\beta\gamma'}$ are on the line $b_{\beta\phi\mu}$. Thus for each fixed β, ($\beta = 1,..., n$), X and Y are on a unique common block of the form $z_{\beta\phi\mu}$, i.e., X, Y are on n common blocks of $\mathbf{D}(n)$.

THEOREM 2. $\mathbf{D}(n)$ *admits a strong tactical decomposition.*

Proof. For each α let the points of $\mathbf{A}_\alpha$ form a point class, and for each fixed pair θ, ϕ we let the blocks of the form $z_{\theta\phi\mu}$ form a class which we denote by $\mathbf{C}_{\theta\phi}$. Clearly there are $n + 1$ point classes and $n(n + 1)$ block classes. Thus if we have defined a tactical decomposition it is certainly strong.

The block class $\mathbf{C}_{\theta\phi}$ consists of those blocks whose images under λ_θ form the parallel class $\mathbf{B}_{\theta\phi}$ of $\mathbf{B}_\theta$ (excluding, of course, the line of $\mathbf{B}_{\theta\phi}$ which contains B_θ). Thus, by (1), each block of $\mathbf{C}_{\theta\phi}$ contains no points of $\mathbf{A}_\phi$ and $n - 1$ points of every other point class. Similarly each point of $\mathbf{A}_\beta$ is on no blocks of any class $\mathbf{C}_{\alpha\beta}$ but on exactly one block of every other class. Hence the point and block classes given form a tactical decomposition.

Remark 1. There are clearly, in general, many nonisomorphic $\mathbf{D}(n)$. The affine planes $\mathbf{A}_\alpha$, $\mathbf{B}_\alpha$ are not assumed to have any specific properties other than their respective orders. So one could replace any of the given planes by another of the same order and obtain another $\mathbf{D}(n)$.

Remark 2. The only intersection numbers of $\mathbf{D}(n)$ are 0, $n - 1$, n. However, all three values definitely occur and so, since strongly resolvable designs only have two intersection numbers, (see [4]), $\mathbf{D}(n)$ is never strongly resolvable.

Remark 3. The construction can obviously be generalized by using incidence structures other than affine planes. (A close inspection shows that we never need all the properties of the affine planes.) However, we need the affine planes to get the strong tactical decomposition, which is why we restrict to this special case.

Remark 4. Note that two blocks are in the same class of the strong tactical decomposition if and only if they have no common point. Further, the points not on any block of a given block class, form a point class. Thus the decomposition is uniquely determined by the design. One immediate consequence of this is that the decomposition must be preserved by all automorphisms of $\mathbf{D}(n)$.

3. THE EMBEDDING

Let $\mathbf{E}_1, \ldots, \mathbf{E}_{n-1}$ be $n-1$ identical copies of an affine plane of order n. (Thus the $\mathbf{E}_k$ are all defined on the same point set $\mathbf{E}$ of n^2 points, and a subset of n points forms a line of $\mathbf{E}_1$ if and only if it forms a line of each $\mathbf{E}_l$, $1 \leqslant l \leqslant n-1$.) Let $\mathbf{E}_{jk}$, $(1 \leqslant j \leqslant n+1)$, denote the parallel classes of $\mathbf{E}_k$ and let e_{ijk}, $(1 \leqslant i \leqslant n)$, be the lines of $\mathbf{E}_{jk}$. Then, clearly, for fixed i, j, the lines e_{ijk}, $(1 \leqslant k \leqslant n-1)$ are the same as point sets.

We now consider an incidence structure $\mathbf{S}(n)$ whose points are the union of the points of $\mathbf{D}(n)$ and $\mathbf{E}$ and whose blocks (considered as point sets) are

(i) the point set $\mathbf{E}$,

(ii) $d_{ijk} = z_{ijk} \cup e_{ijk}$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n+1,\ 1 \leqslant k \leqslant n-1)$.

THEOREM 3. *$\mathbf{S}(n)$ is a symmetric $2-(n^3-n+1, n^2, n)$ design.*

Proof. Clearly the number of points and blocks is n^3-n+1 and each block contains n^2 points.

If $X, Y \in \mathbf{D}(n)$ then there are n blocks of $\mathbf{D}(n)$ containing them. Each of these "extends" to a unique block of $\mathbf{S}(n)$ which contains them and these are the only blocks of $\mathbf{S}(n)$ to which they belong. Hence X, Y are on exactly n common blocks of $\mathbf{S}(n)$. Similarly if $X, Y \in \mathbf{E}$, then X, Y are on exactly one line from each of the $n-1$ affine planes $\mathbf{E}_k$. Each of these $n-1$ lines extends to a block of $\mathbf{S}(n)$ which contains them and so, since $\mathbf{E}$ is also a block, two points of $\mathbf{E}$ are also on n common blocks of $\mathbf{S}(n)$.

Suppose $X \in \mathbf{D}(n)$ and $Y \in \mathbf{E}$. Since $X \in \mathbf{D}(n)$ there is a unique s with $X \in \mathbf{A}_s$. Thus for any i, $(1 \leqslant i \leqslant n)$, and any j, $(1 \leqslant j \leqslant n+1, j \neq s)$, there is a unique k, $(1 \leqslant k \leqslant n-1)$ such that $X \in z_{ijk}$. However for any k, j $(1 \leqslant k \leqslant n-1, 1 \leqslant j \leqslant n+1)$ there is a unique i with $Y \in e_{ijk}$. Hence, for any $j \neq s$, there is a unique pair i, k such that $X \in z_{ijk}$ and $Y \in e_{ijk}$, i.e., for each $j \neq s$ there is a unique block d_{ijk} containing X and Y. Since there are n possibilities for j, this proves the result.

Having constructed $\mathbf{S}(n)$ we can now consider the residual with respect to any other block. Clearly this is always a 2-design with the same parameters as $\mathbf{D}(n)$. However we now show for $n \geqslant 3$ that it is never isomorphic to $\mathbf{D}(n)$ unless the block chosen is $\mathbf{E}$.

THEOREM 4. *If $n \geqslant 3$ and if x is any block of* $\mathrm{S}(n)$ *other than* $\mathbf{E}$ *then the residual with respect to x is not isomorphic to* $\mathbf{D}(n)$.

Proof. Writing S for $\mathrm{S}(n)$ we will prove this result by showing that S^x has an intersection number not equal to 0, $n-1$ or n.

Let $x = d_{ijk}$ and let $y = d_{abc}$ with $b \neq j$. Since $b \neq j$, $|e_{ijk} \cap e_{abc}| = 1$ and hence, as $|x \cap y| = n$, $|z_{ijk} \cap z_{abc}| = n-1$. Let $P = e_{ijk} \cap e_{abc}$ and let $Q \in z_{ijk} \cap z_{abc}$. If g is any integer with $1 \leqslant g \leqslant n-1$ $g \neq j, b$, then (from the proof of Theorem 3) there is a block $z = d_{fgh}$ containing P and Q. Since z_{fgh} contains points from every $\mathbf{A}_\alpha$ $\alpha \neq g$ whereas $z_{ijk} \cap z_{abc}$ contains points from the $\mathbf{A}_\alpha$ with $a \neq j, b$, $|z_{ijk} \cap z_{abc} \cap z_{fgh}| \leqslant n-2$. Thus $2 \leqslant |x \cap y \cap z| \leqslant n-1$ which means that, in S^x, $1 \leqslant |y \cap z| \leqslant n-2$.

Remark 1. When choosing the block z we assumed $n \geqslant 3$. If $n = 2$ then, regarding a point as a trivial affine plane of order 1, S(2) is a $2-(7, 4, 2)$ and the theorem is false.

Remark 2. The embedding of Theorem 3 is not unique, i.e., a given $\mathbf{D}(n)$ does not in general determine a unique $\mathrm{S}(n)$. Clearly the given affine planes $\mathbf{E}_i$ could be replaced by any others of the same order.

COROLLARY. *The full automorphism group of* $\mathrm{S}(n)$ *must fix* $\mathbf{E}$.

REFERENCES

1. R. E. BLOCK, On the orbits of collineation groups, *Math. Z.* **96** (1967), 33–49.
2. H. BEKER, An orbit theorem for designs, *Geometriae Dedicata*, to appear.
3. P. DEMBOWSKI, "Finite Geometries" Springer–Verlag, New York/Berlin, 1968.
4. R. HARRIS, On automorphisms and resolutions of designs, Ph. D. Thesis, University of London, 1974.